

# The Annihilating Ideal of the Fisher Integral

Tamio Koyama

March 19, 2015

## Abstract

In this paper, we discuss a system of differential equations for the Fisher integral on the special orthogonal group. Especially, we explicitly give a set of linear differential operators which generates the annihilating ideal of the Fisher integral, and we prove that the annihilating ideal is a maximal left ideal of the ring of differential operators with polynomial coefficients. Our proof is given by a discussion concerned with an annihilating ideal of a Schwartz distribution associated with the Haar measure on the special orthogonal group. We also give differential operators annihilating the Fisher integral for the diagonal matrix by a new approach.

*Keywords.* Fisher integral, Haar measure, Special orthogonal groups, Annihilating ideal,

MSC classes: 33E20, 16S32

## 1 Introduction

We denote by  $SO(n)$  the special orthogonal group with size  $n$ , i.e.,

$$SO(n) = \{y \in \mathbf{R}^{n \times n} \mid y^\top y = e, \det y = 1\}.$$

Here,  $y^\top$  is the transpose of  $y$  and  $e$  is the identity matrix. The Haar measure  $\mu$  on  $SO(n)$  is a probability measure on  $SO(n)$  which satisfies the equation

$$\int_{SO(n)} f(a^\top y) \mu(dy) = \int_{SO(n)} f(y) \mu(dy)$$

for arbitrary continuous function  $f$  on  $SO(n)$  and any  $a \in SO(n)$ . The Fisher integral on the special orthogonal group is an integral with  $n \times n$  matrix parameter  $x$  given by

$$\int_{SO(n)} \exp(\operatorname{tr}(xy)) \mu(dy) = \int_{SO(n)} \exp\left(\sum_{i,j=1}^n x_{ij} y_{ij}\right) \mu(dy).$$

The Fisher integral is the normalizing constant of the Fisher distribution which is discussed in [9]. Numerical calculations of the normalizing constant is

very important in the point of view of applications in statistics and the holonomic gradient method (HGM) has been applied to such problems. For example, see [7], [4], [10], and [5]. In order to apply the HGM, we need theoretical consideration of differential equations for each problem. In the case of the Fisher integral, [9] gives a system of differential equations for the Fisher integral and they conjectured that the system induces a holonomic ideal. We prove their conjecture positively in this paper. Furthermore, we also prove that this holonomic ideal is an maximal left ideal of a Weyl algebra and consequently it is the annihilating ideal of the Fisher integral.

In order to show these results, we consider a Schwartz distribution which is associated with the Haar measure on the special orthogonal group. We obtain a generating set of the annihilating ideal for the Fisher integral from the annihilating ideal of the Schwartz distribution.

In the application to statistics, differential equations for the Fisher integral in the diagonal case is more important. Actuary, we need a holonomic system for the function

$$f(x_1, \dots, x_n) = \int_{SO(n)} \exp\left(\sum_{i=1}^n x_i y_{ii}\right) \mu(dy).$$

In [9], a system of differential equations for  $f(x)$  was obtained from a differential equation for the matrix hypergeometric function  ${}_0F_1$ . It is also proved that differential operators

$$(x_i^2 - x_j^2) \partial_{x_i} \partial_{x_j} - (x_j \partial_{x_i} - x_i \partial_{x_j}) - (x_i^2 - x_j^2) \partial_{x_k} \quad ((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2))$$

annihilates  $f(x)$  in the case where  $n = 3$ . We give these system of differential equations by a new approach in our paper.

The construction of this paper is as follows. In Section 2, we review the basic notions of Weyl algebra and give some lemmas, which we need in the later section. In Section 3, we give a definition of a distribution associated with the Haar measure on  $SO(n)$ . And we give an generating set of the annihilating ideal of the distribution. In Section 4, we give an generating set of the annihilating ideal of the Fisher integral. We also give a system differential equations for the Fisher integral in the diagonal case by a new approach.

## 2 Weyl Algebra

In this section, we review basic notions in the theory of algebraic analysis, and give a lemma concerning characteristic varieties and maximal ideals which we need for a calculation in the later section.

In the first, we review some notions in algebraic geometry. Let  $n$  be a natural number. We denote by  $x = (x_1, \dots, x_n)$  the standard coordinate of the affine space  $X := \mathbf{C}^n$ . Let  $\mathbf{C}[x] := \mathbf{C}[x_1, \dots, x_n]$  be the polynomial ring with variables  $x_1, \dots, x_n$ . A subset of the space  $X$  is called *an algebraic set* if it can be written as

$$\mathbf{V}(f_1, \dots, f_k) := \{a = (a_1, \dots, a_n) \in \mathbf{X} : f_1(a) = \dots = f_k(a) = 0\}$$

by finite polynomials  $f_1, \dots, f_k \in \mathbf{C}[x]$ . Any ideal  $I$  of the polynomial ring  $\mathbf{C}[x]$  defines an algebraic set:

$$\mathbf{V}(I) := \{a = (a_1, \dots, a_n) \in \mathbf{X} : f(a) = 0, f \in I\} \quad (1)$$

On the other hand, for any algebraic set  $V \subset \mathbf{C}^n$ , we can obtain an ideal of  $\mathbf{C}[x]$  by

$$\mathbf{I}(V) := \{f \in \mathbf{C}[x] : f(x) = 0, x \in V\}. \quad (2)$$

An algebraic set  $V$  is said to be irreducible if it satisfies the following property: if there exist two algebraic set,  $V_1$  and  $V_2$ , such that  $V = V_1 \cup V_2$ , then we have  $V = V_1$  or  $V = V_2$ .

For an algebraic set  $V$ , the *Krull dimension* of  $V$  is the supremum of the length  $k$  of strictly increasing sequence of irreducible algebraic sets such that

$$V_1 \subsetneq \dots \subsetneq V_k \subset V$$

For any ideal  $I \subset \mathbf{C}[x]$ , the Krull dimension of  $\mathbf{V}(I)$  is equals to the degree of the Hilbert polynomial of  $I$  (for example, see e.g. [6]). Algorithms computing Krull dimensions for given algebraic set are given in [3]. In the section 3 and 4, we utilize the methods for computing Krull dimension in this book.

In the next, we review the basic notions of the Weyl algebra. Let us consider the ring of partial differential operators with polynomial coefficients  $D_X := \mathbf{C}\langle x_i, \partial_i : i = 1, \dots, n \rangle$ . Here, we put  $\partial_i := \partial/\partial x_i$  ( $i = 1, \dots, n$ ). It is also called the Weyl algebra. The Weyl algebra  $D_X$  naturally acts on the space of the smooth functions on  $X$ , the space of the Schwartz distributions on  $X$ , and so on. When  $f$  is a smooth function or a Schwartz distribution on  $X$ , we denote by  $p \bullet f$  the function which is given by applying a differential operator  $p \in D_X$  on  $f$ . We call a left ideal

$$\{p \in D_X : p \bullet f = 0\}$$

in  $D_X$  the annihilating ideal of  $f$ , and denote it by  $\text{Ann}(f)$ .

Any element  $p$  of  $D_X$  can be written uniquely as the form of a finite sum  $p = \sum c_{\alpha\beta} x^\alpha \partial^\beta$  ( $c_{\alpha\beta} \in \mathbf{C}$ ). Here,  $\alpha, \beta \in \mathbf{Z}_{\geq 0}^n$  are multi indices, and  $x^\alpha \partial^\beta = \prod_{i=1}^n x_i^{\alpha_i} \partial_i^{\beta_i}$ . For multi index  $\beta \in \mathbf{Z}_{\geq 0}^n$ , we put  $|\beta| = \sum_{i=1}^n \beta_i$ . For a differential operator  $p = \sum c_{\alpha\beta} x^\alpha \partial^\beta \in D_X$ , we put an element  $\text{in}_{(0,1)}(p)$  of a polynomial ring  $\mathbf{C}[x, \xi] := \mathbf{C}[x_i, \xi_i : i = 1, \dots, n]$  as

$$\text{in}_{(0,1)}(p) = \sum_{|\beta|=m} c_{\alpha\beta} x^\alpha \xi^\beta \quad (m = \max\{|\beta| : c_{\alpha\beta} \neq 0\}).$$

For a left ideal  $I$  of  $D_X$ , we call an ideal of  $\mathbf{C}[x, \xi]$  defined by

$$\{\text{in}_{(0,1)}(p) : p \in I\}$$

the *characteristic ideal* of  $I$ , and denote it by  $\text{in}_{(0,1)}(I)$ .

The *characteristic variety* of  $I$  is the algebraic set  $\mathbf{V}(\text{in}_{(0,1)}(I)) \subset X \times \Xi$  defined by the characteristic ideal  $\text{in}_{(0,1)}(I)$ . Here,  $\Xi := \mathbf{C}^n$  and we denote by  $\xi = (\xi_1, \dots, \xi_n)$  the standard coordinate system of the space  $\Xi$ . When  $I \neq D_X$ , the Krull dimension of the characteristic variety of  $I$  is not less than  $n$  (The Bernstein inequality [1],[2]). When the Krull dimension of the characteristic variety equals to  $n$ , the left ideal  $I$  is said to be holonomic.

For calculations in the later sections, we prepare the following lemma concerning with characteristic varieties and maximal ideals:

**Lemma 1.** *If left ideals  $I$  and  $J$  of  $D_X$  satisfies  $I \subsetneq J$ , then we have  $\text{in}_{(0,1)}(I) \subsetneq \text{in}_{(0,1)}(J)$ .*

*Proof.* In this proof, we utilize the theory of Gröbner basis (for example, see e.g., [3]). Let polynomials  $p_1, \dots, p_k$  form a Gröbner basis of  $I$  with respect to the order  $< := <_{(0,1)}$ . Since the left ideal  $J$  is strictly larger than  $I$ , we can take  $p \in J - I$ . Since  $p$  is not included in  $I$ , we obtain the remainder  $r \neq 0$  after division of  $p$  by  $p_1, \dots, p_k$ . By replacing  $p$  to the remainder  $r$ , we can assume  $\text{in}_{<}(p) \notin \text{in}_{<}(I)$  without loss of generality. On the other hand, we have  $\text{in}_{<}(p) \in \text{in}_{<}(J)$  by  $p \in J$ . Therefore,  $\text{in}_{<}(J)$  is strictly larger than  $\text{in}_{<}(I)$ . Suppose  $\text{in}_{(0,1)}(I) = \text{in}_{(0,1)}(J)$ . For arbitrary  $f \in \text{in}_{<}(J)$ , there exists  $p \in J$  such that  $\text{in}_{<}(p) = f$ . Since we have  $\text{in}_{(0,1)}(p) \in \text{in}_{(0,1)}(J) = \text{in}_{(0,1)}(I)$ , there exists  $q \in I$  such that  $\text{in}_{(0,1)}(q) = \text{in}_{(0,1)}(p)$ . Here, we have

$$\begin{aligned} \text{in}_{<}(q) &= \text{in}_{<}(\text{in}_{(0,1)}(q)) \\ &= \text{in}_{<}(\text{in}_{(0,1)}(p)) \\ &= \text{in}_{<}(p) = f. \end{aligned}$$

This contradict that  $\text{in}_{<}(J)$  is strictly larger than  $\text{in}_{<}(I)$ . Therefore, we have  $\text{in}_{(0,1)}(I) \subsetneq \text{in}_{(0,1)}(J)$ .  $\square$

**Lemma 2.** *Suppose a left ideal  $I \subset D_X$  is holonomic. Then  $I$  is a maximal left ideal of  $D_X$  if  $\text{in}_{(0,1)}(I)$  is a prime ideal.*

*Proof.* This proof will be by contradiction. Suppose the left ideal  $I \subset D_X$  is not maximal, then we have  $I = D_X$  or there exists a left ideal  $J$  such that  $I \subsetneq J \subsetneq D_X$ . In the case of  $I = D_X$ , the characteristic variety is the emptyset. This contradict that  $I$  is a holonomic ideal. Hence, the left ideal  $J$  exists. By lemma 1,  $\text{in}_{(0,1)}(J)$  is strictly larger than  $\text{in}_{(0,1)}(I)$ . Since  $\text{in}_{(0,1)}(I)$  is a prime ideal,  $\sqrt{\text{in}_{(0,1)}(I)}$  equals to  $\text{in}_{(0,1)}(I)$ . And this implies

$$\sqrt{\text{in}_{(0,1)}(I)} = \text{in}_{(0,1)}(I) \subsetneq \text{in}_{(0,1)}(J) \subset \sqrt{\text{in}_{(0,1)}(J)}.$$

Hence, we have  $\sqrt{\text{in}_{(0,1)}(I)} \subsetneq \sqrt{\text{in}_{(0,1)}(J)}$ . By the Hilbert's Strong Nullstellensatz, we have  $\mathbf{I}(\mathbf{V}(\text{in}_{(0,1)}(I))) \subsetneq \mathbf{I}(\mathbf{V}(\text{in}_{(0,1)}(J)))$ . Here, we use the notations of (1) and (2). By [3, Chapter 1, Section 4, Proposition 8],  $\mathbf{V}(\text{in}_{(0,1)}(I)) \supsetneq \mathbf{V}(\text{in}_{(0,1)}(J))$  holds. Let  $V$  and  $W$  be the characteristic varieties  $D_X/I$  and  $D_X/J$  respectively. By the above arguments, we have  $V \supsetneq W$ . Since  $\text{in}_{(0,1)}(I)$

is a prime ideal,  $V$  is an irreducible algebraic set. By the Bernstein inequality, the Krull dimension of  $W$  equals to  $n$ . Hence, there exist irreducible algebraic sets  $W_i (i = 1, \dots, n)$  such that  $W_1 \subsetneq \dots \subsetneq W_n \subset W$ . Adding  $V$  to the sequence, we obtain a strictly increasing sequence  $W_1 \subsetneq \dots \subsetneq W_n \subsetneq V$  of irreducible algebraic sets with length  $n + 1$ . However, this contradicts that the dimension of  $V$  equals to  $n$ .  $\square$

The Fourier transformation  $\mathcal{F}$  (resp. the inverse Fourier transformation  $\mathcal{F}^{-1}$ ) for differential operators is a morphism of  $\mathbf{C}$ -algebra from  $D_X$  to  $D_X$  defined by

$$\begin{aligned}\mathcal{F}(x_i) &= -\partial_i, & \mathcal{F}(\partial_i) &= x_i, \\ \mathcal{F}^{-1}(x_i) &= \partial_i, & \mathcal{F}^{-1}(\partial_i) &= -x_i.\end{aligned}$$

Since the Fourier transformation is an isomorphism of  $\mathbf{C}$ -algebra, we have the following lemma:

**Lemma 3.** *If a left ideal  $I \subset D_X$  is maximal, then*

$$\mathcal{F}(I) = \{\mathcal{F}(p) \mid p \in I\}$$

and

$$\mathcal{F}^{-1}(I) = \{\mathcal{F}^{-1}(p) \mid p \in I\}$$

are also maximal left ideals of  $D_X$ .

### 3 Distribution associated with Haar Measure

In this section, we review the Haar measure on the special orthogonal groups, and define a Schwartz distribution associated with this Haar measure. Let  $n$  be a natural number,  $Y$  be a set consisting of  $n \times n$  matrices whose components are real numbers. For  $1 \leq i, j \leq n$ , let  $y_{ij}$  be a function from  $Y$  to  $\mathbf{R}$ . For each point  $y$  in  $Y$ ,  $y_{ij}$  corresponds to  $(i, j)$ -component of  $y$ . The functions  $y_{ij}$  give a local coordinate system of  $Y$ .

The following relations define a submanifold of  $Y$ :

$$\begin{aligned}y^\top y &= e \\ \det y &= 1,\end{aligned}$$

Here,  $y^\top$  denotes the transpose of  $y$  and  $e$  denotes the identity matrix. By the product of matrices, this submanifold defines a Lie group. This Lie group is called the special orthogonal group, and denoted by  $SO_n$ .

On the special orthogonal groups, there uniquely exists the measure  $\mu$  which satisfies the following properties:

$$\begin{aligned}\int_{SO_n} f(y) \mu(dy) &= \int_{SO_n} f(z^\top y) \mu(dy) \quad f \in C^\infty(SO_n), z \in SO_n \\ \int_{SO_n} \mu(dy) &= 1\end{aligned}$$

Here, we denote by  $C^\infty(SO_n)$  the set of continuous functions on  $SO_n$ . We call the measure  $\mu$  the Haar measure on the special orthogonal group.

Let us define a Schwartz distribution on the space  $Y$  associated with the Haar measure  $\mu$  on  $SO_n$ . We denote by  $C_0^\infty(Y)$  the set of continuous functions on  $Y$  with compact supports. For a function  $f$  on  $Y$ ,  $f|_{SO_n}$  denotes the restriction of  $f$  to  $SO_n$ . The map from the functional space  $C_0^\infty(Y)$  to  $\mathbf{R}$  defined by

$$\varphi \mapsto \int f|_{SO_n}(y)\mu(dy) \quad (\varphi \in C_0^\infty(Y))$$

gives a Schwartz distribution on  $Y$ . We denote this distribution by the same notation  $\mu$ .

We denote by  $D_Y := \mathbf{C}\langle y_{ij}, \partial_{ij} : 1 \leq i, j \leq n \rangle$  the ring of differential operators with polynomial coefficient with variable  $y_{ij}$  ( $1 \leq i, j \leq n$ ). Here, we put  $\partial_{ij} := \partial/\partial y_{ij}$ . The annihilating ideal  $\text{Ann}(\mu)$  of the distribution  $\mu$  on  $Y$  is a left ideal of  $D_Y$ . In this section, we explicitly give a set of differential operators which generates the annihilating ideal  $\text{Ann}(\mu)$ . The first step for this purpose is giving some differential operators which annihilate the distribution  $\mu$ . The second step is studying the properties the ideal  $I$  generated by these differential operators. By these properties, we prove  $I = \text{Ann}(\mu)$ .

**Lemma 4.** *The following differential operators annihilate  $\mu$ .*

$$\sum_{k=1}^n (y_{ki}\partial_{kj} - y_{kj}\partial_{ki}) \quad (1 \leq i < j \leq n) \quad (3)$$

$$\delta_{ij} - \sum_{k=1}^n y_{ki}y_{kj}, \quad \delta_{ij} - \sum_{k=1}^n y_{ik}y_{jk} \quad (1 \leq i \leq j \leq n) \quad (4)$$

$$1 - \det y \quad (5)$$

Here,  $\delta_{ij}$  is the kronecker's symbol.

*Proof.* Let  $\varphi$  be a smooth function on  $Y$  with compact support. Since the functions (4) vanish on  $SO_n$ , we have

$$\begin{aligned} \left\langle \left( \delta_{ij} - \sum_{k=1}^n y_{ki}y_{kj} \right) \mu, \varphi \right\rangle &= \left\langle \mu, \left( \delta_{ij} - \sum_{k=1}^n y_{ki}y_{kj} \right) \varphi \right\rangle \\ &= \int_{SO_n} \left( \delta_{ij} - \sum_{k=1}^n y_{ki}y_{kj} \right) \varphi(y) \mu(dy) = 0. \end{aligned}$$

Hence, the differential operator (4) annihilates  $\mu$ . Analogously, we can prove (5) annihilates  $\mu$ .

Let  $E_{ij}$  ( $1 \leq i < j \leq n$ ) be a  $n \times n$  matrix whose  $(k, \ell)$  element is  $\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}$ , and  $c(t) = \exp(tE_{ij})$  for  $t \in \mathbf{R}$ . For a smooth function  $f(y)$  on  $Y$ , we denote  $R_{c(t)}f(y) = f(y \cdot c(t))$ . Let  $v_{ij}$  be a vector field on  $Y$  defined as

$$(v_{ij})_y f = \frac{\partial R_{c(t)}f}{\partial t} \Big|_{t=0} (y) \quad (y \in Y, f \in C^\infty(Y)).$$

It is easy to show that

$$v_{ij} = \sum_{k=1}^n (y_{ki} \partial_{kj} - y_{kj} \partial_{ki}). \quad (6)$$

Note that the differential operator  $\partial_{ij} = \partial/\partial y_{ij}$  can be regarded as a vector field on  $Y$ . Since the measure  $\mu$  is right invariant under  $SO_n$ , we have

$$\begin{aligned} \left\langle \sum_{k=1}^n (y_{ik} \partial_{jk} - y_{jk} \partial_{ik}) \mu, \varphi \right\rangle &= - \left\langle \mu, \sum_{k=1}^n (y_{ik} \partial_{jk} - y_{jk} \partial_{ik}) \varphi \right\rangle \\ &= - \int_{SO_n} (v_{ij} \varphi)(y) \mu(dy) \\ &= - \int_{SO_n} \frac{\partial R_{c(t)} \varphi}{\partial t} \Big|_{t=0} (y) \mu(dy) \\ &= - \lim_{t \rightarrow 0} \int_{SO_n} \frac{\varphi(y \cdot c(t)) - \varphi(y)}{t} \mu(dy) \\ &= 0. \end{aligned}$$

Hence, the differential operator (3) annihilates  $\mu$ .  $\square$

Let  $I$  be an ideal generated by the differential operators (3), (4), and (5). By lemma 4, we have  $I \subset \text{Ann}(\mu)$ . For the opposite inclusion, it is enough to prove the following proposition:

**Proposition 1.** *The left ideal  $I$  is a holonomic ideal, and the characteristic ideal of  $I$  is a prime ideal.*

In fact, by this proposition and lemma 2, the left ideal  $I$  is a maximal ideal of  $D_Y$ . Since  $\text{Ann}(\mu) \neq D_Y$ , we have  $I = \text{Ann}(\mu)$ .

Let  $J$  be an ideal of the polynomial ring  $\mathbf{C}[y, \xi] := \mathbf{C}[y_{ij}, \xi_{ij} : 1 \leq i, j \leq n]$  generated by (4), (5) and

$$\sum_{k=1}^n (y_{ki} \xi_{kj} - y_{kj} \xi_{ki}) \quad (1 \leq i < j \leq n).$$

Obviously,  $J \subset \text{in}_{(0,1)}(I)$  holds and we have  $\mathbf{V}(J) \supset \mathbf{V}(\text{in}_{(0,1)}(I))$ . Now, let us suppose  $J$  is a prime ideal and the Krull dimension of  $\mathbf{V}(J)$  equals to  $n \times n$ . By the Bernstein's inequality, the Krull dimension of  $\mathbf{V}(\text{in}_{(0,1)}(I))$  is not less than  $n \times n$ . Then, we have  $\mathbf{V}(J) = \mathbf{V}(\text{in}_{(0,1)}(I))$ . By the Strong Nullstellensatz, we have  $\sqrt{J} = \sqrt{\text{in}_{(0,1)}(I)}$ . Here, utilizing the assumption that  $J$  is prime, we have  $J = \sqrt{J}$ . Consequently, we have  $J = \sqrt{\text{in}_{(0,1)}(I)} \supset \text{in}_{(0,1)}(I)$ . Hence,  $J = \text{in}_{(0,1)}(I)$  holds. This shows that the Krull dimension of  $\mathbf{V}(\text{in}_{(0,1)}(I))$  equals to  $n \times n$ , i.e., the left ideal  $I$  is holonomic and the ideal  $\text{in}_{(0,1)}(I)$  is prime.

In order to prove proposition 1, it is enough to show the following two statement:  $J$  is a prime ideal and the Krull dimension of  $\mathbf{V}(J)$  equals to  $n \times n$ .

For this purpose, we define an ideal  $J'$  such that  $\mathbf{C}[y, \xi]/J \cong \mathbf{C}[y, \xi]/J'$ , and show that  $J'$  is prime and the Krull dimension of  $\mathbf{V}(J')$  equals to  $n \times n$ .

Let  $J'$  be an ideal of  $\mathbf{C}[y, \xi]$  generated by (4), (5), and

$$\xi_{ij} - \xi_{ji} \quad (1 \leq i < j \leq n). \quad (7)$$

**Lemma 5.** *The quotient ring  $\mathbf{C}[y, \xi]/J$  is isomorphic to  $\mathbf{C}[y, \xi]/J'$  as  $\mathbf{C}$ -algebra.*

*Proof.* Define  $\mathbf{C}$ -algebra homomorphisms  $\phi : \mathbf{C}[y, \xi] \rightarrow \mathbf{C}[y, \xi]$  and  $\psi : \mathbf{C}[y, \xi] \rightarrow \mathbf{C}[y, \xi]$  as

$$\begin{aligned} \phi(y_{ij}) &= y_{ij}, & \phi(\xi_{ij}) &= \sum_{k=1}^n y_{ik} \xi_{kj} \quad (1 \leq i, j \leq n), \\ \psi(y_{ij}) &= y_{ij}, & \psi(\xi_{ij}) &= \sum_{k=1}^n y_{ki} \xi_{kj} \quad (1 \leq i, j \leq n). \end{aligned}$$

By some calculations, we can prove the following formula:

$$\begin{aligned} \phi \left( \sum_{k=1}^n (y_{ki} \xi_{kj} - y_{kj} \xi_{ki}) \right) &= \xi_{ij} - \xi_{ji} - \sum_{\ell=1}^n \left( \delta_{i\ell} - \sum_{k=1}^n y_{ki} y_{k\ell} \right) \xi_{\ell j} \\ &\quad + \sum_{\ell=1}^n \left( \delta_{j\ell} - \sum_{k=1}^n y_{kj} y_{k\ell} \right) \xi_{\ell i} \end{aligned} \quad (8)$$

$$\psi(\xi_{ij} - \xi_{ji}) = \sum_{k=1}^n (y_{ki} \xi_{kj} - y_{kj} \xi_{ki}) \quad (9)$$

$$\phi\psi(\xi_{ij}) = \xi_{ij} - \sum_{\ell=1}^n \xi_{\ell j} \left( \delta_{i\ell} - \sum_{k=1}^n y_{ki} y_{k\ell} \right) \quad (10)$$

$$\psi\phi(\xi_{ij}) = \xi_{ij} - \sum_{\ell=1}^n \xi_{j\ell} \left( \delta_{i\ell} - \sum_{k=1}^n y_{ik} y_{\ell k} \right) \quad (11)$$

Let  $p_1 : \mathbf{C}[y, \xi] \rightarrow \mathbf{C}[y, \xi]/J$  ( $p_1(f) = \bar{f}$ ) and  $p_2 : \mathbf{C}[y, \xi] \rightarrow \mathbf{C}[y, \xi]/J'$  ( $p_2(f) = \bar{f}$ ) be the projections, then we have  $\ker(p_2\phi) = J$  and  $\ker(p_1\psi) = J'$ . In fact,  $\ker(p_2\phi) \supset J$  follows by (8), and  $\ker(p_1\psi) \supset J'$  follows by (9). Let  $f \in \ker(p_2\phi)$ , then we have  $\phi(f) \in J'$ . By (9), we have  $\psi\phi(f) \in J$ . Since we also have  $f - \psi\phi(f) \in J$  by (11),  $f$  is an element of  $J$ . Hence,  $\ker(p_2\phi) = J$  holds. Analogously, if we take  $f \in \ker(p_1\psi)$ , then  $\psi(f)$  is an element of  $J$ . The equation (8) implies  $\psi\phi(f) \in J'$ . The equation (10) also implies  $f - \phi\psi(f) \in J'$ , and we have  $f \in J'$ . Hence,  $\ker(p_1\psi) = J'$  holds also.

By the isomorphism theorem, we have two morphisms,  $\mathbf{C}[y, \xi]/J \rightarrow \mathbf{C}[y, \xi]/J'$  and  $\mathbf{C}[y, \xi]/J' \rightarrow \mathbf{C}[y, \xi]/J$ . We can show by some calculations that their compositions equal to the identity morphisms.  $\square$

In order to prove that the ideal  $J'$  is prime, we utilize a tensor product of  $\mathbf{C}$ -algebras. The following lemma is well known:



**Lemma 6.** Let  $\mathbf{C}[x] := \mathbf{C}[x_1, \dots, x_n]$ ,  $\mathbf{C}[y] := \mathbf{C}[y_1, \dots, y_m]$ , and  $\mathbf{C}[x, y] := \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_m]$  be polynomial rings. And we denote by  $\iota_1 : \mathbf{C}[x] \rightarrow \mathbf{C}[x, y]$  and  $\iota_2 : \mathbf{C}[y] \rightarrow \mathbf{C}[x, y]$  the immersion maps. Let  $I_1$  and  $I_2$  be ideals of  $\mathbf{C}[x]$  and  $\mathbf{C}[y]$  respectively. Then, there exists the following isomorphism:

$$\mathbf{C}[x]/I_1 \otimes_{\mathbf{C}} \mathbf{C}[y]/I_2 \cong \mathbf{C}[x, y]/I \quad \left( \overline{f} \otimes \overline{g} \mapsto \overline{\iota_1(f)\iota_2(g)} \right)$$

where  $I = \mathbf{C}[x, y]\iota_1(I_1) + \mathbf{C}[x, y]\iota_2(I_2)$ .

*Proof.* see, e.g., [6, I, §6, Proposition 1].  $\square$

Now, let us compute the ideal  $J'$ .

**Lemma 7.** The ideal  $J'$  is a prime ideal and the Krull dimension of  $\mathbf{V}(J')$  equals to  $n \times n$ .

*Proof.* In the first, we calculate the dimension of  $\mathbf{V}(J')$ . Let  $J'_1$  be an ideal of  $\mathbf{C}[y_{ij} : 1 \leq i, j \leq n]$  generated by the polynomials (4) and (5). The algebraic set  $\mathbf{V}(J'_1)$  equals to the special orthogonal group, and its Krull dimension is  $n(n-1)/2$ . Moreover, the ideal  $J'_1$  is prime by [11, p147, Theorem(5.4c)]. Especially, the algebraic set  $\mathbf{V}(J'_1)$  is irreducible.

Let  $J'_2$  be an ideal of  $\mathbf{C}[\xi_{ij} : 1 \leq i, j \leq n]$  generated by the polynomials (7). Let  $<$  be a graded lexicographic order which satisfies  $\xi_{ij} > \xi_{ji}$  ( $1 \leq i < j \leq n$ ). The polynomials (7) form a Gröbner basis of  $J'_2$  with respect to the order  $<$ . Hence, the Krull dimension of  $\mathbf{V}(J'_1)$  equals to  $n(n+1)/2$ . Besides, the quotient ring  $\mathbf{C}[\xi_{ij} : 1 \leq i, j \leq n]/J'_2$  is isomorphic to a polynomial ring  $\mathbf{C}[\xi_{ij} : 1 \leq i \leq j \leq n]$ . In fact, let  $\varphi : \mathbf{C}[\xi_{ij} : 1 \leq i, j \leq n]/J'_2 \rightarrow \mathbf{C}[\xi_{ij} : 1 \leq i \leq j \leq n]$  and  $\psi : \mathbf{C}[\xi_{ij} : 1 \leq i \leq j \leq n] \rightarrow \mathbf{C}[\xi_{ij} : 1 \leq i, j \leq n]/J'_2$  be morphisms defined by

$$\varphi(\overline{\xi}_{ij}) = \begin{cases} \xi_{ij} & (i \leq j) \\ \xi_{ji} & (i > j) \end{cases}, \quad \psi(\xi_{ij}) = \overline{\xi}_{ij},$$

then  $\varphi\psi$  and  $\psi\varphi$  are isomorphisms. Since  $\mathbf{C}[\xi_{ij} : 1 \leq i \leq j \leq n]$  is an integral domain,  $\mathbf{C}[\xi_{ij} : 1 \leq i, j \leq n]/J'_2$  is also an integral domain. Hence,  $J'_2$  is a prime ideal.

By  $\mathbf{V}(J') = \mathbf{V}(J'_1) \times \mathbf{V}(J'_2)$ , the Krull dimension of  $\mathbf{V}(J')$  equals  $n(n-1)/2 + n(n+1)/2 = n^2$ .

In the second, we show that the ideal  $J'$  is prime. Since  $\mathbf{V}(J'_1)$  and  $\mathbf{V}(J'_2)$  are irreducible algebraic sets, their product  $\mathbf{V}(J') = \mathbf{V}(J'_1) \times \mathbf{V}(J'_2)$  is irreducible also. Hence, the coordinate ring  $\mathbf{C}[y, \xi]/\sqrt{J'}$  of  $\mathbf{V}(J')$  is an integral domain. Also, we have an isomorphism between the coordinate rings:

$$\mathbf{C}[y, \xi]/\sqrt{J'} \rightarrow \mathbf{C}[y]/J'_1 \otimes_{\mathbf{C}} \mathbf{C}[\xi]/J'_2 \quad \left( \overline{f(y)g(\xi)} \mapsto \overline{f(y)} \otimes \overline{g(\xi)} \right).$$

By lemma 6, we have an isomorphism

$$\mathbf{C}[y]/J'_1 \otimes_{\mathbf{C}} \mathbf{C}[\xi]/J'_2 \rightarrow \mathbf{C}[x, \xi]/J' \quad \left( \overline{f(y)} \otimes \overline{g(\xi)} \mapsto \overline{f(y)g(\xi)} \right).$$

These isomorphisms give an isomorphism

$$\mathbf{C}[y, \xi]/\sqrt{J'} \rightarrow \mathbf{C}[y, \xi]/J' \quad \left( \overline{f(y)g(\xi)} \rightarrow \overline{f(y)g(\xi)} \right).$$

Since this isomorphism implies that  $\mathbf{C}[y, \xi]/J'$  is an integral domain,  $J'$  is a prime ideal.  $\square$

Therefore, we have the following theorem.

**Theorem 1.** *The differential operators (3), (4), and (5) generate the annihilating ideal of the Schwartz distribution  $\mu$  associated with the Haar measure on the special orthogonal group.*

## 4 The Fisher Integral

In this section, we use the notations in Section 3. Especially,  $\mu$  denotes the Haar measure on the special orthogonal group. For a square matrix  $a$ , we put  $\text{etr}(a) := \exp(\text{tr}(a))$ .

The following lemma is useful [8]:

**Lemma 8.** *Let  $D_n := \mathbf{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$  be the ring of differential operators with polynomial coefficients. Let  $u$  be a Schwartz distribution and  $f$  be a polynomial in  $\mathbf{C}[x_1, \dots, x_n]$ . We put  $f_i := \partial f / \partial x_i$ . Suppose a left ideal  $I$  of  $D_n$  is holonomic and annihilates  $u$ . Then, the left ideal  $J$  generated by*

$$\{P(x_1, \dots, x_n; \partial_{x_1} - f_1, \dots, \partial_{x_n} - f_n) | P(x_1, \dots, x_n; \partial_{x_1}, \dots, \partial_{x_n}) \in I\}$$

*is holonomic and annihilates the distribution  $e^f u$ .*

The Fisher integral is the following function defined by an integration on the special orthogonal group:

$$f(x) = \int_{SO(n)} \text{etr}(xy) \mu(dy) \quad (x \in \mathbf{C}^{n \times n}).$$

Here,  $x = (x_{ij})$  is an  $n \times n$  matrix over the field of real numbers. In this section, we explicitly give the annihilating ideal of the Fisher integral  $f(x)$  as an application of Theorem 1.

In [9], it is proved that the Fisher integral  $f(x)$  annihilated by the following differential operators:

$$\sum_{k=1}^n (x_{ki} \partial_{x_{kj}} - x_{kj} \partial_{x_{ki}}) \quad (1 \leq i < j \leq n) \quad (12)$$

$$\delta_{ij} - \sum_{k=1}^n \partial_{x_{ki}} \partial_{x_{kj}}, \quad \delta_{ij} - \sum_{k=1}^n \partial_{x_{ik}} \partial_{x_{jk}} \quad (1 \leq i \leq j \leq n) \quad (13)$$

$$1 - \det \partial_x, \quad (14)$$

Here, we denote by  $\det \partial_x$  the determinant of the matrix whose  $(i, j)$ -element is  $\partial_{x_{ij}}$ .

**Theorem 2.** *The annihilating ideal of the Fisher integral  $f(x)$  is generated by the differential operators (12), (13), and (14).*

*Proof.* Let  $I$  be a left ideal generated by the differential operators (12), (13), and (14). Since the ideal  $I$  equals to  $\mathcal{F}^{-1}(\text{Ann}(\mu))$  and  $\text{Ann}(\mu)$  is maximal,  $I$  is also maximal by Lemma 3. By  $\text{Ann}(f) \neq D_X$  and the maximality of  $I$ , we have  $I = \text{Ann}(f)$ .  $\square$

**Corollary 1.** *The left ideal generated by (12), (13), and (14) is a maximal ideal of  $D_X$ . And this ideal is a holonomic ideal of  $D_X$ .*

In application to statistics, the case where the matrix  $x$  is diagonal is more important. In this case, the Fisher integral is a function with respect to  $x_1, \dots, x_n$ , which are the diagonal elements of  $x$ , defined by

$$\tilde{f}(x_1, \dots, x_n) = \int_{SO(n)} \exp\left(\sum_{i=1}^n x_i y_{ii}\right) \mu(dy). \quad (15)$$

A system of differential equations for (15) was given in [9].

**Proposition 2** ([9]). *The differential operator*

$$\partial_{x_i}^2 - \sum_{k \neq i} \frac{1}{x_i^2 - x_k^2} (x_i \partial_{x_i} - x_k \partial_{x_k}) - 1 \quad (i = 1, \dots, n) \quad (16)$$

*annihilates (15).*

When  $n = 3$ , there are extra differential operators annihilating (15).

**Proposition 3** ([9]). *When  $n = 3$ , the differential operators*

$$(x_i^2 - x_j^2) \partial_{x_i} \partial_{x_j} - (x_j \partial_{x_i} - x_i \partial_{x_j}) - (x_i^2 - x_j^2) \partial_{x_k} \quad ((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2))$$

*annihilates (15).*

As an application of Theorem 1, we give new proofs for Proposition 2 and Proposition 3.

*Proof of Proposition 2.* By Lemma 8, the integrand  $\exp(\sum_{i=1}^n x_i y_{ii}) \mu(dy)$  is annihilated by

$$\begin{aligned} p_{ij} &:= \sum_{k=1}^n (y_{ki} \partial_{y_{kj}} - y_{kj} \partial_{y_{ki}}) - y_{ji} x_j + y_{ij} x_i \quad (1 \leq i < j \leq n), \\ \tilde{p}_{ij} &:= \sum_{k=1}^n (y_{ik} \partial_{y_{jk}} - y_{jk} \partial_{y_{ik}}) - y_{ij} x_j + y_{ji} x_i \quad (1 \leq i < j \leq n), \\ q_i &:= \partial_{x_i} - y_{ii} \quad (1 \leq i \leq n). \end{aligned}$$

Here, we regard  $\exp(\sum_{i=1}^n x_i y_{ii}) \mu(dy)$  as a Schwartz distribution. Considering the elements of

$$\frac{1}{x_i^2 - x_j^2} \begin{pmatrix} x_i & x_j \\ x_j & x_i \end{pmatrix} \begin{pmatrix} p_{ij} \\ \tilde{p}_{ij} \end{pmatrix}$$

for  $1 \leq i < j \leq n$ , we have that the differential operators

$$\begin{aligned} & y_{ij} + \frac{1}{(x_i + x_j)(x_i - x_j)} \left( x_i \sum_{k=1}^n (y_{ki} \partial_{y_{kj}} - y_{kj} \partial_{y_{ki}}) + x_j \sum_{k=1}^n (y_{ik} \partial_{y_{jk}} - y_{jk} \partial_{y_{ik}}) \right), \\ & y_{ji} + \frac{1}{(x_i + x_j)(x_i - x_j)} \left( x_j \sum_{k=1}^n (y_{ki} \partial_{y_{kj}} - y_{kj} \partial_{y_{ki}}) + x_i \sum_{k=1}^n (y_{ik} \partial_{y_{jk}} - y_{jk} \partial_{y_{ik}}) \right) \end{aligned}$$

annihilate the integrand. Since the differential operator  $-1 + \sum_{j=1}^d y_{ij}^2$  annihilates the integrand, the differential operator

$$\begin{aligned} & -1 + \partial_{x_i}^2 - \sum_{j \neq i} \frac{y_{ij}}{x_i^2 - x_j^2} \left( x_i \sum_{k=1}^n (y_{ki} \partial_{y_{kj}} - y_{kj} \partial_{y_{ki}}) + x_j \sum_{k=1}^n (y_{ik} \partial_{y_{jk}} - y_{jk} \partial_{y_{ik}}) \right) \\ & = -1 + \partial_{x_i}^2 - \sum_{j \neq i} \frac{1}{x_i^2 - x_j^2} \left( x_i \sum_{k=1}^n (y_{ki} \partial_{y_{kj}} - y_{kj} \partial_{y_{ki}}) + x_j \sum_{k=1}^n (y_{ik} \partial_{y_{jk}} - y_{jk} \partial_{y_{ik}}) \right) y_{ij} \\ & \quad + \sum_{j \neq i} \frac{1}{x_i^2 - x_j^2} (x_i y_{ii} - x_j y_{jj}) \end{aligned}$$

also annihilates the integrand. Hence, we have that the operator (16) annihilates (15).  $\square$

Our proof of Proposition 3 utilizes the following well-known formula: for  $n \times n$  regular matrix  $A$  and  $I, J \subset [n]$  with  $|I| = |J|$ , we have

$$[A^{-1}]_{I,J} = \det A^{-1} [A]_{I',J'}. \quad (17)$$

Here,  $I', J'$  denote the subsets of indices complementary to  $I, J$ , and  $[A]_{I,J}$  denotes the minor of  $A$  corresponding to  $I, J$ .

*Proof of Proposition 3.* We can assume that  $(i, j, k) = (1, 2, 3)$  without loss of generalities. Let  $I = \{1, 2\}$  and  $J = \{3\}$ . Then the equation (17) implies that the all  $3 \times 3$  special orthogonal matrix  $A = (a_{ij})$  satisfies the equation

$$\det \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \det (a_{33}).$$

Hence, the operator

$$y_{11}y_{22} - y_{12}y_{21} - y_{33}$$

annihilates the integrand  $\exp(\sum_{i=1}^n x_i y_{ii}) \mu(dy)$ . By the analogous way to the proof of Proposition 2, we have that the differential operator

$$\begin{aligned} & \partial_{x_1} \partial_{x_2} + \frac{y_{12}}{x_2^2 - x_1^2} \left( x_2 \sum_{k=1}^3 (y_{k2} \partial_{y_{k1}} - y_{k1} \partial_{y_{k2}}) + x_1 \sum_{k=1}^3 (y_{2k} \partial_{y_{1k}} - y_{1k} \partial_{y_{2k}}) \right) - \partial_{x_3} \\ &= \partial_{x_1} \partial_{x_2} + \frac{1}{x_2^2 - x_1^2} \left( x_2 \sum_{k=1}^3 (y_{k2} \partial_{y_{k1}} - y_{k1} \partial_{y_{k2}}) + x_1 \sum_{k=1}^3 (y_{2k} \partial_{y_{1k}} - y_{1k} \partial_{y_{2k}}) \right) y_{12} - \partial_{x_3} \\ & \quad + \frac{1}{x_2^2 - x_1^2} (y_{11} x_2 - y_{22} x_1) \end{aligned}$$

annihilates the integrand. Hence, the differential operator

$$\partial_{x_1} \partial_{x_2} + \frac{1}{x_2^2 - x_1^2} (x_2 \partial_{x_1} - x_1 \partial_{x_2}) - \partial_{x_3}$$

annihilates the integral (15).  $\square$

## References

- [1] J. E. Björk. *Rings of differential operators*. North-Holland, New York, 1979.
- [2] S. C. Coutinho. *A Primer of Algebraic D-modules*. Number 33 in London Mathematical Society Student Texts. Cambridge University Press, 1995.
- [3] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms*. Springer, 1992.
- [4] H. Hashiguchi, Y. Numata, N. Takayama, and A. Takemura. Holonomic gradient method for the distribution function of the largest root of Wishart matrix. *Journal of Multivariate Analysis*, 117:296–312, 2013.
- [5] T. Koyama, H. Nakayama, K. Nishiyama, and N. Takayama. Holonomic gradient descent for the fisher-bingham distribution on the  $d$ -dimensional sphere. *Computational Statistics*, 29:661–683, 2014.
- [6] David Mumford. *The Red Book of Varieties and Schemes*. Springer, Berlin Heidelberg, second edition, 1999.
- [7] H. Nakayama, K. Nishiyama, M. Noro, K. Ohara, T. Sei, N. Takayama, and A. Takemura. Holonomic gradient descent and its application to the Fisher-Bingham integral. *Advances in Applied Mathematics*, 47:639–658, 2011.
- [8] T. Oaku, Y. Shiraki, and N. Takayama. Algorithms for d-modules and numerical analysis. In Z. Li and W. Sit, editors, *Computer mathematics*, pages 23–39, River Edge, 2003. World Scientific.

- [9] T. Sei, H. Shibata, A. Takemura, K. Ohara, and N. Takayama. Properties and applications of Fisher distribution on the rotation group. *Journal of Multivariate Analysis*, 116:440–455, 2013.
- [10] T.Sei and A. Kume. Calculating the normalising constant of the bingham distribution on the sphere using the holonomic gradient method. *Statistics and Computing*, 2013.
- [11] Hermann Weyl. *The classical groups: their invariants and representations*. Princeton University Press, 1946.